Advanced Insurance Play in 21: Risk Aversion and Composition Dependence

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Abstract. Risk-averse strategies for card counting in blackjack or twenty-one were first introduced in the landmark paper of Friedman (1980). These strategies are based upon optimizing the player's utility and/or reducing the player's long-run index (variance divided by square of expectation) rather than merely optimizing expectation. This concept has now become part of mainstream blackjack theory and has been endorsed in the popular literature (Schlesinger 2005). There are commercially available programs that will now compute risk-averse indices for all plays, with the notable exception of the most important one, insurance. Risk-averse strategy for this play necessarily involves the concept of partial insurance. This was presented in Griffin (1999) for the very important special case of insuring a natural.

This paper will give a comprehensive treatment of the theory of risk-averse insurance. We will present and derive the pertinent mathematical equations and show how to compute optimal partial insurance as well as insurance indices. This necessarily involves the production of different indices for different hand totals, and thus requires a discussion of composition-dependent indices.

1 Introduction

In the game of blackjack (twenty-one), players are offered "insurance" whenever the dealer's up-card is an Ace. In most casinos, players may wager up to half their original bet, and this bet is paid off at 2 to 1 if the casino holds a ten-valued card under its Ace, giving it a natural. It is promoted as an opportunity to "insure" a good hand. In reality, it usually functions as simply a side bet that the dealer's hole card is a 10. (This paper will actually be an exception to that, in that here we will explore the "insurance" aspects of the insurance wager.) Since approximately 4/13 of the cards in a full pack are 10s, a simple calculation shows that the expectation on this bet is approximately $3(4/13) - 1 = -1/13 = -7.6\%$, making it one of the worst wagers in the casino. Wagers like this are sometimes called "sucker bets." However, a card-counting strategy is obvious: If more than 1/3 of the remaining cards are 10s, then the wager has positive expectation for the player. This option is quite valuable for a card counter, and all modern counting systems have an index number for the insurance play. Indeed, this is the most important index number in most systems.

Indices in counting systems have traditionally been based solely on maximizing the player's expectation. However, in recent years the more appropriate concept of so-called "risk-averse" indices has become popular among blackjack players. This is a playing strategy based upon maximizing the player's utility, and minimizing her long-run index: variance divided by expectation squared. The concept was originally introduced by Joel Friedman in his landmark paper (Friedman 1980), and now has become part of mainstream blackjack theory. It has recently been endorsed in the popular literature (Schlesinger 2005) and most commercial simulators will now produce risk-averse index numbers for every playing decision except insurance.

Insurance is inherently a different and more complicated type of decision because there is (negative) correlation between this bet and the other wager the player has made. Most players have an intuitive understanding that insuring a "good hand" reduces variance, and that these hands should be insured at somewhat below the index. To some extent, the opposite is true for a weak hand. So the proper strategy will depend on the player's holding. Also, risk aversion becomes more important with bigger bets, so a risk-averse insurance strategy will depend on the size of our bets. We should note that our strategy will have more application to players who wager on two spots, as this will produce larger optimal bets.

There was actually some preliminary discussion of this topic by Peter Griffin in his definitive book *The Theory of Blackjack* (1999). Griffin computes the strategy for insuring the best possible hand, a blackjack, that should be used by a Kelly player, i.e., one using logarithmic utility. He shows that the optimal strategy includes taking *partial insurance* under certain circumstances, which will be the key to our discussion. Griffin's solution is exact for this special case, but a more general treatment will require approximations.

My interest in this issue arose after Internet discussions with Steven Heston in the early days of the `bj21.com` website. In particular, he pointed out to me the crucial role played by partial insurance and he helped to derive some of the formulas for p_v below.

2 Risk-averse playing strategy

If advantage players simply wished to maximize their expectations, they would risk their entire fortunes on each positive-expectation bet. Although there is an infinitesimal probability that such a player will become immensely wealthy, the overwhelming probability is that he/she will eventually go bankrupt. This is not usually regarded as a rational approach by responsible authors. Instead, most texts on blackjack or advantage play advocate "optimal betting" strategies that seek to maximize a utility function. Such a function allows quantification of the risk-reward tradeoff. For blackjack purposes, we can approximate a wager's certainty equivalent (CE) for reasonable utility functions with the quadratic[1]

$$\mathrm{CE} \approx \mathrm{EV} - \frac{1}{2}\,\frac{\mathrm{Var}}{k\,\mathrm{Bank}}, \tag{1}$$

where EV is the expected value of the wager, Var is its variance, and Bank is the size of the player's bank, with EV, $\sqrt{\mathrm{Var}}$, and Bank measured in monetary units (e.g., dollars). The parameter k is a constant that depends on the player's utility function. It reflects the player's risk aversion/risk tolerance. For logarithmic utility, $k = 1$. This utility gives rise to the famous "Kelly" betting system.

Note that to apply this formula we don't need to know either the player's Bank or her risk aversion k, but only the product $k\,\mathrm{Bank}$. This is sometimes called the "Kelly-equivalent" bank.

Given a particular wagering opportunity, we can compute its mean μ and standard deviation σ expressed as proportions of the amount wagered. σ^2 is the variance, which we will denote by v. Let f be the amount wagered, expressed as a fraction of the Kelly-equivalent bank. Then our certainty equivalent becomes

$$\mathrm{CE} \approx k\,\mathrm{Bank}\left(\mu f - \frac{vf^2}{2}\right) \tag{2}$$

This function is maximized by selecting a betting fraction $f = \mu/v$, which is the basic equation for optimal betting. Note that optimal betting could be called "risk-averse betting."

Conventional playing strategy in blackjack simply looked at the expected value of each decision and selected the one that was highest. Risk-averse strategy looks at the certainty equivalent of each decision and selects the one whose CE is highest. It should really be called "optimal" playing strategy, if we use that term in the same away that we do when we speak of "optimal betting."

3 Underlying mathematics

Throughout the rest of the paper we will employ the following notation:

- p — The proportion of insurance that we take. Full insurance is wagering one-half of our bet on the insurance line and corresponds to $p = 1$. In most casinos, p is constrained to be between 0 and 1.
- d — The density of 10-valued cards remaining in the pack.
- μ — The expectation of the insurance wager, namely $3d - 1$.

[1] A derivation of this formula is provided in Appendix 2.

R The result of the playing hand when the dealer has a natural. With standard rules, this is 0 if we have a blackjack and -1 for other hands.

CondEV The conditional expectation of the playing hand, computed under the assumption that the dealer does not have a BJ.

Suppose that we are in an insurance situation and we take partial insurance p. This partial insurance will affect both our expected value and our variance. We can compute this as a function of p, and then obtain our overall certainty equivalent. Our CE will be a quadratic function of p, and it will obtain a maximum value for the optimal level insurance p_{opt}. After some mathematical work, the details of which are found in Appendix 1, we obtain the following simple and elegant formula for p_{opt}:

$$p_{\text{opt}} = p_0 + p_v, \tag{3}$$

where

$$p_0 = \frac{2}{9}\left[\frac{3d-1}{d(1-d)}\right]\frac{1}{f} = \frac{2\mu}{f(2+\mu-\mu^2)} \approx \frac{\mu}{f}, \tag{4}$$

$$p_v = \frac{2}{3}[-R + \text{CondEV}]. \tag{5}$$

Each of these terms has a special significance, which we will elaborate on below. The first term, p_0, represents the optimal insurance bet we would make in a stand-alone situation, where we did not have any money wagered on a BJ hand. It tells us the amount we would bet if we had an opportunity to make an "over-the-shoulder" insurance bet on another player's hand. The amount of this wager in monetary units does not depend on f; its approximate value is simply $(\mu/2)(k\,\text{Bank})$. Eq. (4) expresses this insurance bet in terms of the amount of partial insurance to take on a hand with a blackjack wager of size $f \cdot (k\,\text{Bank})$. The p_v term minimizes the overall variance of the result of our insurance bet and our playing hand.

Our formula for p_v is exact. However we will use an approximation for p_0. Our primary interest in risk-averse partial insurance will be when the magnitude of the expectation $|\mu|$ on the insurance wager is small. Otherwise we will either not take insurance, or take as much insurance as we have available, depending on the algebraic sign of μ. Our simplification will be valid for small values of μ.

Under ordinary blackjack rules, R is -1 except when the player has a blackjack. For these hands $p_v = (2/3)[1 + \text{CondEV}]$. But if the player holds a blackjack, $R = 0$ and $\text{CondEV} = 3/2$, in which case $p_v = 1$, as we would expect. We summarize:

$$p_v = \begin{cases} 1 & \text{if player holds a blackjack,} \\ (2/3)[1+\text{CondEV}] & \text{if player holds any other hand.} \end{cases} \tag{6}$$

This applies for standard blackjack rules. If early surrender or European hole card rules are applicable, then we would have to use a different value for R.

4 Minimizing variance with p_v

Let us consider the case where $d = 1/3$ and our insurance expectation is 0. Whatever insurance wager we make will have no effect on our overall expectation. It will affect our variance, and our optimal strategy is clearly to minimize the variance. We do that by taking $p = p_v$ specified above. Note that the value of p_v depends entirely on our playing hand. Strong hands that have high CondEV will require higher insurance bets. Weak hands will require smaller insurance bets.

We estimated the CondEV for each possible playing hand. We put together an infinite pack that reflected the composition that a 10-counter would see at the break-even point for insurance ($d = 1/3$). A combinatorial analysis was performed over that pack. We assumed that the dealer stands on soft 17. The values are listed in the Table A1 of the Appendix, along with the corresponding values of p_v.

Our formula calls for full insurance if we have a natural. This is obviously correct for by doing so we guarantee that we make "even money" no matter what the dealer holds and we have reduced our variance to 0! But a blackjack is not the most insurable hand. The table shows that for a 20, we minimize variance by taking 111% insurance. The reader may be surprised because 20 is a weaker hand than 21. But with a blackjack, there is no possibility of a loss, and so there is less risk to avert.

As an exercise, the reader may wish to consider a variation on blackjack in which the house takes ties on blackjack. Then $R = -1$, and CondEV $= 3/2$, so our formula calls for $(2/3)[1 + 3/2] = 140\%$ of insurance, higher than that for a 20.

Of course, as a practical matter most casinos do not allow a player to over-insure, so if we held a 20, we would merely take the sub-optimal full insurance.

The reader may be surprised to see that even with a poor hand, some partial insurance is called for. An easy example to look at is if the player has a hand that she wishes to surrender. Our formula calls for taking 1/3 insurance. That is, if the player wagered 6 units, our formula calls for a 1-unit insurance wager. Consider what happens. If the dealer has a BJ, we lose our 6 units but make 2 units on insurance, for a net loss of 4 units. But if the dealer doesn't have BJ, we lose 1 unit on the insurance and 3 units on the surrender. Again the total loss is 4 units. We always lose the same 4 units; our variance has been reduced to 0.

This helps illustrate why we always take some insurance. Even a weak hand has some value. Insuring that small value reduces our variance.

It is customary for players who take full insurance on a blackjack to simply request "even money." If late surrender and partial insurance are both available, the player could just as simply ask for the return of "one-third money" of the wager on poor hands, effectively offering to surrender two-thirds. I will refer to this maneuver as "modified early surrender" (MES) since it is mathematically equivalent to a type of surrender that could be done before the dealer checks for a blackjack. However, I would advise the reader against this, since it may indicate some mathematical understanding of the insurance bet, which is undesirable given the current state of paranoia among casino personnel.

For slot machines, it is customary to measure the expected value in terms of the amount of money returned by the machine. That is, a machine with EV of -3% is said to return 97%. This return is $1+\text{CondEV}$. Note that this is the term that is our formula for p_v: It calls for us to insure 2/3 of the payback.

For another exercise, consider an unusual rule variation where a certain player hand is an automatic loser. That is, its $\text{CondEV} = -1$. This particular hand returns 0 and our formula yields $p_v = 0$. This would be the extreme case in which we would have nothing to insure.

5 Rounding strategies and playing two hands

Of course, we cannot make exact insurance bets; we have to do some rounding. Our formula is based upon a quadratic approximation, and quadratics are symmetric around their vertices. This means that "round to the nearest" will give us the best possible result. For example, if an insurance bet of 1.3 units is optimal, it would be better to bet 1 unit rather than 2. But if 1.6 units were optimal, it would be better to wager 2 units.

What if the player can take only full insurance? If the choice is "all or nothing," then our critical value is 50%. Take full insurance when $p > 50\%$, and do not insure when $p < 50\%$. We can set $p_v = 0.5$ and solve to obtain $\text{CondEV} = -25\%$. At the break-even point, we would insure hands whose CondEV is above this value, and decline insurance for the others. From our table, this includes hard 8 and better hands. In particular, we do not take full insurance on a stiff. (To our knowledge, the first discussion of this strategy was by Peter Griffin 1988.)

A corollary of this is that taking full insurance on a stiff hand actually increases the overall variance.

Now what if the player is betting on two spots, and so has two hands in action? We could treat these two hands as one entity with a certain aggregate CondEV and an aggregate R. Then we would plug these into our formula for p_v. However, CondEV is linear and R is linear (actually R is just the conditional expectation under the assumption that there is a dealer BJ) our final result will just be the average of the p_v values of the individual hands, weighted by bet size.

Here is an example. Suppose we bet 12 units on the left spot and 6 units on the right. Inevitably, we get a blackjack on the right and a 16 on the left. Assume that we have a break-even situation for the insurance expectation. Our table tells us to take full insurance on the 6 unit bet (bet 3 units) , but take 1/3 insurance on the 12 unit hand (bet 2 units). Our total insurance bet should be 5 units. We have taken 5/9 of our available insurance. This is the weighted average of 1/3 and 1, with the 1/3 weighted twice.

Examples like this have led most blackjack players to prefer wagers of equal amounts when they are playing 2 spots. This is optimal for reducing variance. If we are betting equal amounts, then we just do a simple average.

Even if the casino requires "all or nothing" insurance, we can insure one hand and not the other. Effectively, this gives us half-insurance. If the decision is "all, half, or none," then our critical values are 25% and 75%. If less than 25% of insurance is optimal, then we take no insurance. Between 25% and 75%, we insure one of our two spots. Above that, we insure both.

Even a worse hand (16) calls for 30% insurance. If we have two such hands, our average is above 25%, so we should insure one of them. So we conclude: If we are playing two hands and we have a break-even insurance situation, we always insure at least one of them.

If the hands are strong enough, we can take full insurance on both. We need the sum of their p_v values to be 150%, so that the average is 75%. From Table A1 in Appendix 2, we see that the following combinations are sufficiently strong.

$$\begin{aligned} 20 &+ \text{hard 12 or better,} \\ \text{BJ} &+ \text{hard 8 or better,} \\ 19 &+ \text{hard 9 (borderline) or better,} \\ \text{AA or } 11 &+ \text{hard 10 or better.} \end{aligned}$$

Of course, if partial insurance is allowed, there will be times where we take full insurance on one spot and partial insurance on the other. Mathematically, it doesn't matter which spot we put the full insurance on, but for the sake of appearances, I recommend that you put the bigger insurance bet in front of the stronger hand.

Let me again emphasize that all of this discussion has been for the break-even case, $\mu = 0$. In the next section, we discuss the more general situation.

6 The over-the-shoulder term p_0

The other term in our formula is p_0. While p_v depends only on the hand we are playing, p_0 depends on the insurance expectation and the size of our initial bet. The simplest way to understand p_0 is to imagine this situation: Another player has wagered an amount f on a blackjack hand and is faced with an insurance option. He/she doesn't want to take it, but offers to allow you to exercise his/her option. What size insurance bet would you make?

Basically you would use your utility function and compute the "optimal bet" in a 2-to-1 wager game with expectation μ. If you used our quadratic approximation, you would obtain a bet $b = [\mu/(2+\mu-\mu^2)](k\,\text{Bank})$. Now $f = 1/(k\,\text{Bank})$. Also, $p = 2b$. Substituting these, we obtain $p = 2\mu/[(2+\mu-\mu^2)f]$, which was our original approximation for p_0. Of course, we simplified that further by approximating $p \approx \mu/f$.

For a Kelly player who uses logarithmic utility, there is an exact solution to this problem. The optimal bet for a wager that pays 2 : 1 is $\mu\,\text{Bank}/2$. Translating this to our p and f notation gives our $p = \mu/f$. That is, our approximation for p_0 appears to be actually exact for the case of Kelly players. However, this is not completely accurate. While our methods give the correct value for the p_0-term, as we have described it, they do not give the exact answer for the overall value of p_{opt}. This is also discussed in Appendix 1.

The reader may be surprised that we simply add the p_0 and p_v terms together to get our optimal bet. The next example may help to clarify that.

Example 1. Suppose that we hold a natural and the dealer has an Ace up. We have a large wager of $f = 4\%$ and our card-counting system estimates the insurance expectation is -1%. $p_v = 100\%$, and our $p_0 = 1\%/4\% = -25\%$. If it were permitted, our optimal bet would be $1 - 0.25 = 75\%$ insurance. Think of this as a two-stage process. First we put out a 100% insurance bet. This is mathematically equivalent to settling the hand for "even money." Now we have essentially withdrawn the hand; mathematically it is as though we no longer had any money on the betting square. But we have a negative-EV bet in front of us, and our optimal bet on that would be to wager -25% of our allowed insurance bet. Ordinarily casinos do not accept negative insurance bets, but here we can accomplish the same effect by pulling back some of the money that we had planned to place on the insurance line.

Note that this is an example where risk-aversion causes us to make a negative-expectation insurance wager. It also helps to illustrate the relationship between optimal betting and risk-averse play. Declining "even-money" on a natural is essentially making a bet. The size of that bet should be computed proportionally, just like the size of any other bet. If the edge is marginal, we would not make a "big bet"; that is, we would not completely decline the even money.

Example 2. Suppose that we held a 16, and the insurance expectation is $+1\%$. Since we plan to surrender the 16, our p_v is $33\frac{1}{3}\%$, and p_0 is 25%. Our optimal play would be 58% insurance. Again, think of two-stage betting. When we put out a $33\frac{1}{3}\%$ insurance bet, we are essentially settling for the return of one-third of our original wager, effectively surrendering two-thirds, which I earlier called MES. We have removed all the variance from our playing hand, just as if we had no wager on the table. But again, we have a positive-EV bet in front of us on which we should make an optimal wager of p_0.

If our only choice was all or nothing, then we would slightly prefer full insurance. But half-insurance would be better if permitted. Here is a case where we desire to take less than full insurance even though the insurance bet has positive expectation.

Example 3. Same as example 2, except that the insurance edge is only 0.5%. Then p_0 become 12.5% and our optimal play is 46%. If our only choice is "all or nothing" insurance, then we would actually decline insurance, even though it is a positive-EV bet.

Our equations give us a linear relationship between p and μ. The player will usually measure μ via a count system, and our formulas give us p as a linear function of μ. However the typical constraint on p, $0 \le p \le 1$, changes our function into a piecewise linear function, of the form

$$p = \begin{cases} 0 & \text{if } \mu > \mu_l, \\ \mu/f + p_v & \text{otherwise,} \\ 1 & \text{if } \mu < \mu_r, \end{cases} \tag{7}$$

as seen in the graphs in the Appendix.

We can determine the left and right endpoints μ_l and μ_r by taking $p = 0$ and $p = 1$, respectively, in our equations

$$\mu \approx [p - p_v]f. \tag{8}$$

Note that the length of the interval is proportional to the bet size f. The midpoint of this interval (μ for $p = 0.5$) gives us the index for all-or-nothing insurance. The quartiles ($p = 0.25$ and $p = 0.75$) give us the indices for half, all, or nothing. The interval will usually contain the conventional index, $\mu = 0$. The exceptions are 20, where the conventional index lies to the right of the interval, and blackjack, where it is the right endpoint.

We illustrate this by doing the computations for a 19 ($p_v = 80\%$) with a bet size of 2%.

left endpoint:	$\mu_l \approx (0 - 0.8)2\% = -1.6\%$
right endpoint:	$\mu_r \approx (1 - 0.8)2\% = 0.4\%$
"all or nothing" index:	$\mu \approx (0.5 - 0.8)2\% = -0.6\%$
"half or nothing" index:	$\mu \approx (0.25 - 0.8)2\% = -1.1\%$
"half or all" index:	$\mu \approx (0.75 - 0.8)2\% = -0.1\%$

Note that the length of the interval is 2%, the bet size.

In the Appendix, we have included graphs that illustrate the typical partial insurance interval. We have taken d, the density of 10s to be the independent variable. Of course, d is just a linear function of μ, namely $(\mu + 1)/3$.

The first graph is based upon a hard 18, $p_v \approx 0.6$, with a 5% bet. As an exercise the reader may verify that the interesting points have μ-values of -3.0%, -1.8%, -0.5%, 0.8%, 2.0%, and d-values of 32.3%, 32.8%, 33.2%, 33.6%, 34.0%. These are the points that correspond to the p-values of 0, 0.25, 0.5, 0.75, 1.0.

Additional graphs are included to show the effects of hand-strength and bet size.

In practice, the player will employ a count system to estimate d and μ. This will be a linear functions of a parameter called the "true count." With that linear function, we can transform these μ indices into count indices. In the next two sections, we will illustrate this with two popular count systems: the unbalanced 10-count and the Hi-Lo count.

7 Unbalanced 10-count

A 10-count gives a perfect estimation of insurance expectation μ. There are a number of different ways of counting 10s, but in our opinion the simplest is the unbalanced ten count (UTC). In the UTC, we count 10s as -2, and all other cards as $+1$. We start with an initial running count of $-4N$, where N is the number of decks used. In a 6-deck game, we would start at -24.

When the count is 0 (called the pivot point), insurance breaks even ($\mu = 0$). For positive counts, insurance has positive expectation, and for negative counts it has negative expectation. Thus the conventional insurance player can make decisions based simply on the algebraic sign of the UTC. A more subtle feature of the UTC is that the insurance expectation μ is actually equal to its true count when measured in points per card. If instead we employ a true count T of points per deck, then $\mu = T/52$.

We can use this simple relationship to change the μ-indices described above into T-indices, by simply taking $T = 52\mu$. Let us illustrate this by revisiting the example at the end of the previous section: a 19 with a bet size of 2%.

left-endpoint:	$\mu \approx -1.6\%$ so T-index is $(-1.6\%)(52) \approx -0.83$
right-endpoint:	$\mu \approx 0.4\%$ so T-index is $(0.4\%)(52) \approx 0.21$
"all or nothing" index:	$\mu \approx -0.6\%$ so T-index is $(-0.6\%)(52) \approx -0.31$
"half or nothing" index:	$\mu \approx -1.1\%$ so T-index is $(-.1.1\%)(52) \approx -0.57$
"half or all" index:	$\mu \approx -0.1\%$ so T-index is $(-0.1\%)(52) \approx -0.05$

The length of our interval is $(2\%)(52) = 1.04$ points per card. The shifts in indices here will be very moderate and of very limited value. If we had a bigger bet of 5% out, then the length of the interval would be 2.6 points per card, leading to some playable index shifts.

We have tabulated these indices for the 2% and 5% bet sizes in Tables A2 and A3 of the Appendix. These indices are based on playing one hand. However, if the player is wagering on two hands, he/she can simply take the weighted averages of the indices.

The table for 5% bets does give some interesting results for the "all, half, or full" insurance. This part is useful to players who have 2 spots and wish to decide between insuring neither, one, or both spots. Note that for good hands, the index for half-insurance is shifted 1–2. This means that the player should buy some insurance, even though it is a negative-EV bet. Also, the "full insurance" index is shifted to the right for weak hands. If the player has two stiffs, he/she should not insure both of them until the true count is a full point per deck over the index.

8 Another count system: Hi-Lo

While the 10-count is perfect for the insurance wager, it has a weak betting correlation of 0.72. Other counts are usually used for betting purposes. While there are players who have the remarkable ability to keep multiple counts and use them for different purposes, most players keep only one count, and use that count for all decisions. In this section we will show how they could compute our indices and discuss some of the other issues that occur.

Count systems typically determine a parameter called the "true count" T. A linear model may be used to relate deck composition, and therefore insurance expectation, to T. For insurance, our equation has the form:

$$\mu \approx C(T - \text{Index}_0), \tag{9}$$

where Index_0 is the conventional insurance index, and C is a proportionality constant. We can solve this for T and substitute into the previous equation (8) to get our risk-averse index

$$\text{RAIndex} \approx \text{Index}_0 + \frac{(p - p_v)f}{C}. \tag{10}$$

We plug in the critical values for p for the decision we are considering. For "all or nothing insurance," we use $p = 0.5$.

One of the most popular count systems is the Hi-Lo. We will discuss it here, partly because it is so common, but also because it illustrates the problems that will occur with other count systems.

The Hi-Lo treats Aces and 10s as "high cards" and the player counts -1 for each that is seen. Cards 2–6 are considered low and are counted $+1$. The $7, 8, 9$ are "neutral" and ignored (counted as 0). The player divides this "running count" by the number of unseen cards (or decks) to form a "true count." This true count gives a measure of deck composition and can be used to estimate expectation for betting purposes and for playing decisions.

If we were playing from an infinite deck, we could compute our insurance index as follows. Let T_c be the true count in points per card. Let $T_d = 52T_c$ be the points per deck. T_c may be increased by having a higher density of 10s and Aces, or a lower density of 2–6. We treat each of these as equally likely, so a representative pack with true count T_c has $T_c/10$ extra Aces, 10s, Js, Qs, Ks, and $T_c/10$ fewer 2s, 3s, 4s, 5s, 6s. Note that there are 10 ranks that are counted, and four of these are 10-valued. Since the initial 10s density is 4/13, our 10s-density and insurance expectation are approximately

$$d \approx \frac{4}{13} + \frac{4}{10} T_c, \tag{11}$$

$$\mu = 3d - 1 \approx -\frac{1}{13} + \frac{12}{10} T_c \approx -\frac{1}{13} + \frac{12}{520} T_d \approx -7.60\% + 2.30\% T_d. \tag{12}$$

Setting $\mu = 0$ gives us the infinite deck index $T_d = 10/3 \approx 3.33$ which is the "full pack" index for betting that the next card is a 10.

Thus when we compute Hi-Lo indices, we use a factor of 2.3% to translate expectation into indices. The general formula is

$$\text{RAIndex} \approx \text{Index}_0 + \frac{(p - p_v)f}{2.3\%}, \tag{13}$$

where we plug in the critical value for p for the decision that we want, such as 0.5 for "all-or-nothing insurance."

Note that the length of the partial insurance interval is $f/2.3\%$. See Table 1.

Table 1. Length of Hi-Lo partial insurance interval.

bet size	interval length
1%	0.4
2%	0.9
3%	1.3
4%	1.7
5%	2.2
6%	2.6

Fortunately for blackjack players, but unfortunately for blackjack analysts, blackjack is not dealt from an infinite deck. When we make a decision to insure a specific hand, we have some additional information that some specific cards have been removed from the pack. That is, we can compute different indices for each hand composition, which will be more accurate than a generic index.

For Hi-Lo, the known removal of Aces makes a big difference for our insurance decisions. Hi-Lo does not distinguish between the Ace and the 10. If a number of Aces are removed, then insurance is more favorable, but Hi-Lo thinks it is less favorable. We therefore adjust our Hi-Lo index to compensate for this. Similar, but much smaller, effects exist for the other denominations.

Peter Griffin appears to have been the first to have observed this phenomenon. Griffin did a thorough computer analysis and determined exact values for all the single-deck compositions. He documented it in a private letter to Prof. Edward Thorp on October 17, 1970. This was re-published in Griffin (1998) shortly after Griffin's death.

We will use a linear approximation to estimate the composition-dependent indices. Some of the mathematical justification for this will be included in Appendix 1. For each known removal, we will add or subtract an adjustment to our insurance index. Table 2 shows the approximate adjustments.

Table 2. Adjustments to insurance index. (N is the number of decks.)

Ace	$-1.9/N$
neutrals $(7, 8, 9)$	$-0.8/N$
10s	$+0.8/N$
lows (2–6)	$+0.21/N$

Let us illustrate this by computing the generic index for insuring a generic hand for the various deck sizes. We add $-1.9/N$ to adjust for the dealers Ace. See Table 3.

Table 3. Estimating generic Hi-Lo insurance indices.

single deck	$3.3 - 1.9/1 = 1.4$
double deck	$3.3 - 1.9/2 = 2.4$
six decks	$3.3 - 1.9/6 = 3.0$

These are approximately the traditional indices used by Hi-Lo players. They are listed in Wong (1994, Table 11), where they are attributed to Dr. Griffin.

We have seen that risk-averse insurance depends on the specific hand we are playing, and in previous sections we have showed how to shift the conventional index to obtain risk-averse indices. However, each of the specific hands has a different composition-dependent conventional index. For example, the conventional index for $10, 10$ (Hard 20) can be estimated using the method above. For each 10 we add $+0.8/N$ to the "generic indices" above, for a total of $1.6/N$. This gives the results in Table 4.

Note that we are working at cross-purposes to risk-aversion! Risk-aversion raises the index for 20 slightly, but the conventional index should be raised to adjust for the composition.

The situation can be different for other hands. For example, if we have an Ace and 9, the composition adjustment is $(-1.9 - 0.8)/N = -2.7/N$, giving Table 5.

Table 4. Hi-Lo conventional insurance: 10, 10 vs. Ace.

single deck	$1.4 + 1.6/1 = 3.0$	(adjustment of $+1.6$)
double deck	$2.4 + 1.6/2 = 3.2$	(adjustment of $+0.8$)
six decks	$3.3 + 1.6/6 = 3.6$	(adjustment of $+0.3$)

Table 5. Hi-Lo conventional insurance: Ace, 9 vs. Ace.

single deck	$1.4 - 2.7 = -1.3$	(adjustment of -2.7)
double deck	$2.4 - 2.7/2 = +1.1$	(adjustment of -1.3)
six decks	$3.3 + 2.7/6 = +2.9$	(adjustment of $+0.4$)

The same phenomena will occur for other hands. For example, risk-aversion raises the index for 16. But the composition considerations raise the index for 9, 7, although they do lower it for 10, 6.

We should also consider the orders of magnitude of these shifts. For modest sized bets of 2%, the entire partial insurance interval is less than 1 point per deck. The shift in the index is less than that. But in single deck, the composition adjustments are typically greater and sometimes move in the opposite direction. The single-deck player who uses a lower index for insuring 10, 10 will be making an error.

To illustrate this phenomenon, we have included tables for Hi-Lo insurance for one- and six-deck games. Table A4 shows a single-deck game and Table A5 shows a six-deck one. We have also re-computed our estimates of the conditional expectation and the resulting p_v parameter for these specific games, using rules more typical for them. For the single-deck game, we assumed the house hits soft 17 and for the six-deck game, we assumed that it stands on it. (In doing the computations for 8, 8, we assumed that double after split was permitted in the six-deck game but not in the single-deck.) We have shown the *shifts* in the indices for different bet sizes, so that the reader can gauge the order of magnitude of these effects.

9 Bet-sizing and practical considerations

As we have seen, optimal insurance involves a great deal of complexity. There are several parameters that go into each index, and there is a different index for each play. Some simplifications are needed.

First, let us consider the issue of bet-sizing. We should assume that the player is using an “optimal” betting system, as it would be rather strange for someone to consider risk-aversion in insurance while ignoring it in betting! This in turn depends somewhat on the bet spread that the player uses. Usually, players use a much smaller spread at single-deck than at multiple-deck.

At single-deck, bets rarely get over 1% or 2% of the Kelly-equivalent bank. We see that at these levels, risk-aversion is a minor factor. But composition dependence is a very important factor in single-deck. So for single-deck players, the composition-dependent effects are the most important.

For single-deck players that wish to use different indices for various hands, I would recommend they focus on the composition dependence. Small RA-adjustments could be made for one representative bet size, such as 1%. I have included a complete table of single-deck insurance indices for this level in Table A6 of the Appendix. These have indices for all our insurance decisions. At 1%, they are all fairly close to each other, and I would suggest that the player only use one. (Single-deck players would do even better to just learn side-count, which would improve both their insurance decisions and their other playing decisions as well.)

Wider spreads are often used in multiple-deck games. Total bets of 4% of the Kelly-equivalent bank are not that uncommon, particularly for someone playing two spots. In six-deck, the composition effects are minor, but at this bet level the RA-effects are significant. To show this, I have included a table of complete six-deck indices at the 4% bet level, as Table A7. Note that Tables A4/A5 show the shifts in indices, while Tables A6/A7 show the actual indices.

I would suggest that six-deck players start with the principle "When in doubt insure one of two spots." They can refine the play by looking at our "half, all, or none" indices. Note that for the "good hands," half insurance can be taken as low as 2 when there is a big bet out. For poor hands, full-insurance can be delayed until almost 4 when there is a big bet.

Again, if you are playing two spots, you simply average the indices for each hand, weighting them by bet size.

Appendix 1. More mathematical details

In this appendix we will further elaborate on some of the details that were omitted in the foregoing discussion.

A1.1 Optimizing the partial insurance

First, we will provide the derivation of our basic equations for the optimal partial insurance bet. Our total result is the sum of two random variables. One is the result of the insurance wager, and the other is the original playing hand itself. We will temporarily call these Ins and Hand, respectively, and will let $\text{Tot} = \text{Ins} + \text{Hand}$ be the sum of these. The bet on the insurance wager is $p/2$ times the wager f on the hand. Note that $\text{E(Tot)} = \text{E(Ins)} + \text{E(Hand)}$, whereas $\text{Var(Tot)} = \text{Var(Ins)} + \text{Var(Hand)} + 2\,\text{Cov}$, where Cov is short for Cov(Ins, Hand). Our overall certainty equivalent is given by $\text{E(Tot)}f - \text{Var(Tot)}f^2/2$. If we express these in terms of the Ins and Hand variables, there will be five terms, two expectation terms, two variance terms, and the covariance term. Now the E(Hand) and Var(Hand) terms are basically just the certainty equivalent CE(Hand). If we write MCE(Ins) for the marginal certainty equivalent that the insurance bet adds, then we have

$$\mathrm{CE(Tot)} = \mathrm{CE(Hand)} + \mathrm{MCE(Ins)}, \tag{14}$$

where

$$\mathrm{MCE(Ins)} = \left(\frac{pf}{2}\right) \mathrm{E(Ins)} - \frac{1}{2}\left(\frac{pf}{2}\right)^2 \mathrm{Var(Ins)} - \frac{1}{2}\, 2 \left(\frac{pf}{2}\right) f \,\mathrm{Cov}\,. \tag{15}$$

Note that CE(Hand) is a constant term; its value does not depend on the amount p of partial insurance that we take. Essentially, this means that it is a "sunk cost," which we can do nothing about. Note that the Var(Hand) only appears here, and so it has no effect on the optimal value of p in our approximation.

Our equation for MCE(Ins) is a quadratic in p, and we can rewrite it as

$$\mathrm{MCE(Ins)} = \alpha p - \beta p^2, \tag{16}$$

where

$$\alpha = \left(\frac{f}{2}\right) \mathrm{E(Ins)} - \left(\frac{f^2}{2}\right) \mathrm{Cov}, \tag{17}$$

$$\beta = \left(\frac{f^2}{8}\right) \mathrm{Var(Ins)}. \tag{18}$$

Our quadratic will obtain a maximum at its vertex $p_{\mathrm{opt}} = \alpha/(2\beta)$ or

$$p_{\mathrm{opt}} = p_0 + p_v, \tag{19}$$

where

$$p_0 = 2\,\frac{\mathrm{E(Ins)}}{\mathrm{Var(Ins)}}\,\frac{1}{f}, \tag{20}$$

$$p_v = -2\,\frac{\mathrm{Cov}}{\mathrm{Var(Ins)}}. \tag{21}$$

Each of these terms has a special significance. The p_0 term is exactly the optimal insurance bet we would make if we considered insurance as a stand-alone bet, without reference to the hand. Our optimal bet b_{opt} would be $[\mathrm{E(Ins)}/\mathrm{Var(Ins)}]k\,\mathrm{Bank}$. Temporarily let b_H be the amount that we have wagered on the hand, so that $f = b_H/(k\,\mathrm{Bank})$. Then $k\,\mathrm{Bank} = b_H/f$. Since a full insurance wager is $b_H/2$, our optimal p will be $2b/b_H = 2[\mathrm{E(Ins)}/\mathrm{Var(Ins)}]k\,\mathrm{Bank}/b_H = 2[\mathrm{E(Ins)}/\mathrm{Var(Ins)}](b_H/f)/b_H$, which is just (20).

Note also that p_v minimizes the overall variance. If we wrote out the equation for the overall variance, it would contain the Var(Ins) and the Cov term from the MCE(Ins) equation above, plus the Var(Hand) term. But the Var(Hand) term is just constant, and it will not affect the optimal value.

Let us compute the value of p_0. Insurance is just a 2-to-1 bet with probability d of success, so $\mathrm{E(Ins)} = \mu = 3d - 1$ and $\mathrm{Var(Ins)} = 9d(1-d)$ or $2 + \mu - \mu^2$. So this gives us

$$p_0 = \frac{2}{9}\left[\frac{3d-1}{d(1-d)}\right]\frac{1}{f} = \frac{2\mu}{f(2+\mu-\mu^2)} \approx \frac{\mu}{f}. \tag{22}$$

For the p_v term, we calculate the covariance. Consider a representation of the random variable Hand as a vector of possible outcomes $\langle R, X_1, X_2, \ldots, X_n\rangle$ with corresponding probabilities $\langle d, q_1, q_2, \ldots, q_n\rangle$. The vector for insurance is $\langle 2, -1, -1, \ldots, -1\rangle$. Note that $\sum q = 1 - d$ and that $\mathrm{CondEV} = \sum(qX)/(1-d)$.

Now the covariance is $\mathrm{E(Ins \cdot Hand)} - \mathrm{E(Ins)} \cdot \mathrm{E(Hand)}$ and

$$\mathrm{E(Ins \cdot Hand)} = d(2R) + \sum(-1)qX = 2dR - (1-d)\,\mathrm{CondEV}, \tag{23}$$

$$\mathrm{E(Ins)} \cdot \mathrm{E(Hand)} = \mu \cdot [dR + (1-d)\,\mathrm{CondEV}]. \tag{24}$$

Substituting in $\mu = 3d - 1$ and simplifying we obtain

$$\mathrm{Cov} = 3d(1-d)(R - \mathrm{CondEV}), \tag{25}$$

so that

$$p_v = -2\,\frac{\mathrm{Cov}}{\mathrm{Var(Ins)}} = -2\,\frac{3d(1-d)(R-\mathrm{CondEV})}{9d(1-d)} = \frac{2}{3}[-R + \mathrm{CondEV}]. \tag{26}$$

A1.2 Kelly digression: Do two wrongs make a right?

We used a quadratic approximation for CE. The "exact" solution that optimized this approximate function contained the p_0-term of $2\mu/[f(2+\mu-\mu^2)]$. We approximate this as μ/f. Now as pointed out above in the section on p_0, this is exactly the amount of partial insurance that a Kelly player would take on a stand-alone insurance bet. When I first considered this, it appeared that our formula was exact for Kelly players. It looked as if two approximations canceled each other, and that two wrongs made a right.

However, this is not quite true. While our methods give the correct value for the p_0-term, as we have described it, they do not give the exact answer for the overall value of p_{opt}.

A Kelly player uses logarithmic utility. For the case where there is no variance in the playing hand (such as when the player holds a blackjack or plans to surrender) we can actually compute the exact value of p_{opt} as

$$p_{\mathrm{opt}} = \frac{\mu}{f} + \frac{2}{3}[-R + \mathrm{CondEV}] + \mu\left[\frac{R + 2\,\mathrm{CondEV}}{3}\right] \tag{27}$$

or

$$p_{\text{opt}} = p_0 + p_v + \mu \left[\frac{R + 2\,\text{CondEV}}{3} \right]. \tag{28}$$

To derive this, let $X_1 = p + R$ be the amount of our gain if the house has a natural, and $X_2 = -p/2 + \text{CondEV}$ be our gain if it does not. A Kelly player wishes to maximize his/her expected logarithm, which may be expressed

$$\begin{aligned} \mathrm{E}L &= d\ln(1 + X_1 f) + (1-d)\ln(1 + X_2 f) \\ &= d\ln[1 + (p+R)f] + (1-d)\ln[1 + (-p/2 + \text{CondEV})f]. \end{aligned} \tag{29}$$

We wish to find the value of p that maximizes this. We take the derivative of EL with respect to p obtaining

$$(\mathrm{E}L)'(p) = \frac{df}{1 + (p+R)f} - \frac{1}{2}\,\frac{(1-d)f}{1 + (-p/2 + \text{CondEV})f}. \tag{30}$$

We find the optimal p by setting this to 0 and solving. After simplifying the result, we obtain the following:

$$p_{\text{opt}} = \frac{3d-1}{f} + 2d\,\text{CondEV} + (d-1)R. \tag{31}$$

This expresses p_{opt} in terms of d, but we would like to have it in terms of $\mu = 3d - 1$. Note that $d = (\mu + 1)/3$. Our first term is simply μ/f. The second term becomes $(2/3)(\mu + 1)$ CondEV, or $\mu(2/3)$ CondEV $+(2/3)$ CondEV. The final term becomes $[(\mu + 1)/3 - 1]R$ or $(\mu/3)R - (2/3)R$. If we group the two μ-terms together, we simply obtain the last term in (27); if we group the other terms together, we have p_v.

The last term of (28) represents the error in our approximation. It is clearly proportional to μ. For the case where the dealer holds a natural, the error is just $(1)\mu$; for the case of a surrender hand, the error is $(-2/3)\mu$. The case where the player holds a blackjack is worked out in Griffin (1999), where the exact answer is expressed in terms of the parameter that we are calling d, the density of 10s.

There is a simple explanation for this. Recall how we described the "two-step process" above. First, we take partial insurance p_v to minimize variance. If we hold a natural, this means that we take "even money." Then we take the Kelly-optimal stand-alone insurance bet, $\mu\,$Bank. We have approximated Bank as $1/f$, but this is actually our pre-deal Bank. After we take "even money," our Bank has gone up by 1 bet, and is now $1/f + 1$. Thus the Kelly optimal bet is $\mu(1/f + 1)$, as computed from (28). If we had a surrender hand, then our bank would go down by 2/3 of a bet when we execute an MES so our p_0-term should really be $(1)(1/f - 2/3)$. Our quadratic approximation is not clever enough to adjust for the change in our fortune.

This is a very general phenomenon, and it occurs with any utility function, provided the conditional payoffs to our blackjack hand are constant. To see this, let U denote our utility, and let B denote our bank, measured in terms of the number of our current bets. (In other words, assume that our bet is 1 unit). Let $X_1 = p + R$ be the amount of our gain if the house has a natural, and $X_2 = -p/2 + \text{CondEV}$ be our gain if it does not. Then our expected utility is

$$\begin{aligned} \text{E}U &= dU(B + X_1) + (1 - d)U(B + X_2). \\ &= dU(B + p + R) + (1 - d)U(B - p/2 + \text{CondEV}). \end{aligned} \tag{32}$$

Let us introduce a new variable $p' = p - p_v = p - (2/3)[-R + \text{CondEV}]$. The optimal value of p' is the exact value of our p_0.

Now if we accept "even-money" or "third-money" by taking p_v partial insurance, our expected result is $[2\,\text{CondEV} + R]/3$. Call this amount EX. Note that

$$X_1 - \text{E}X = p + R - \frac{1}{3}[2\,\text{CondEV} + R] = p - \frac{2}{3}[-R + \text{CondEV}] = p'. \tag{33}$$

And with a little more algebra,

$$\begin{aligned} X_2 - \text{E}X &= -\frac{p}{2} + \text{CondEV} - \frac{1}{3}[2\,\text{CondEV} + R] \\ &= -\frac{1}{2}\left[p - \frac{2}{3}(\text{CondEV} - R)\right] = -\frac{p'}{2}, \end{aligned} \tag{34}$$

so that our expected utility (32) becomes

$$\text{E}U = dU(B + \text{E}X_1 + p') + (1 - d)U(B + \text{E}X - p'/2). \tag{35}$$

Let $B' = B + \text{E}X$. This our adjusted bank, after we accept "even money" or "early surrender two-thirds." From (33), we see that $B + X_1$ is $B + \text{E}X + p' = B' + p'$. From (34) we see that $B + X_2 = B + \text{E}X - p'/2 = B' - p'/2$. Substituting these into (32) gives us

$$\text{E}U = dU(B' + p') + (1 - d)U(B' - p'/2). \tag{36}$$

This is precisely the same as our expected utility if we were making a stand-alone insurance bet, of size p', from our adjusted bank B'. The value of p' that optimizes this will be precisely the optimal value of p_0 as it was described earlier.

Of course, this relies heavily on the fact that our payoffs are constant. For most hands, they are not constant, and the exact solution would have an additional term reflecting the conditional variance of our hand.

A1.3 Composition-dependent insurance

Here we will give a little more justification for the composition-dependent adjustments that were discussed above in the section on the Hi-Lo true count

T. Throughout this section, T will be measured in points per deck. We begin with the simple case of insurance in a single-deck game.

We can treat this insurance situation as being dealt from a 51-card pack, which has had 1 Ace removed. Because it counted the Ace, the Hi-Lo count starts with an initial running count of -1. The initial insurance 10s-density is 16/51. Now the remaining pack has 39 cards that are counted by Hi-Lo, of which 16 are 10s. This means that our estimate of insurance density is

$$d \approx \frac{16}{51} + \left(\frac{T}{52} - \frac{-1}{51}\right)\frac{16}{39}. \tag{37}$$

Setting $d = 1/3$ and solving, we arrive at the value of $T = 299/204 \approx 1.46$, which is an improved approximation to the conventional insurance index. Note that is about 1.87 points below the conventional index. If we had removed 2 Aces, and started with a 50 card pack, the same analysis would suggest that the second face further lowers the index by 1.93. The third lowers it 1.97, and the fourth lowers it by 2.0. We will use the value of 1.9 as an approximation for the effect of removing Aces on the single deck insurance index.

With more decks, the effect will be proportionally less. In N decks, we could use the equation

$$d \approx \frac{16N}{52N-1} + \left(\frac{T}{52} - \frac{-1}{52N-1}\right)\frac{16N}{40N-1}. \tag{38}$$

Solving this gives us the approximations for conventional indices

$$\text{Ins Index}\,(N \text{ decks}) \approx \frac{-13}{12}\,\frac{160N^2 - 92N + 1}{(-52N+1)N}. \tag{39}$$

Values produced by this formula are shown below.

Table 6. Approximate Hi-Lo insurance indices.

1 deck	$299/204 \approx 1.46$
2 decks	$5941/2472 \approx 2.40$
4 decks	$3029/1116 \approx 2.71$
6 decks	$67717/22392 \approx 3.02$

In our earlier discussion, we had used a simple linear approximation that produced indices of 1.4, 2.4, 2.7, and 3.0 respectively. Our linear approximation is adequate for producing indices that are accurate to one decimal place. Additional accuracy does not offer much practical value.

The reader may check that the numbers in Table 6 have limit 10/3 as N approaches infinity. Our formula gives us the following indices. Indeed, if we express it as a power series in the reciprocal $1/N$ we obtain

$$\text{Ins Index}\,(N \text{ decks}) \approx \frac{10}{3} - \frac{289}{156}\left(\frac{1}{N}\right) - \frac{5}{388}\left(\frac{1}{N}\right)^2 \tag{40}$$

or

$$\text{Ins Index}\,(N \text{ decks}) \approx \frac{10}{3} - (1.85) - (0.15)\left(\frac{1}{N}\right) - (0.15)\left(\frac{1}{N}\right)^2. \tag{41}$$

We will further simplify this by approximating it as $10/3 - 1.9/N$. Note the role played by the number of decks N in computing our index.

There is a similar effect for removing the other cards tracked by Hi-Lo: the 10s, the lows, and the neutrals. In a similar way, the effect is roughly proportional to the reciprocal of the number of decks. We will mercifully spare the reader the details of obtaining the corresponding approximations.

Appendix 2. Derivation of CE equation

The concept of certainty equivalent arises from utility theory. We assume that a person has a utility function $U(B)$, expressed in terms of wealth B. We will assume that U is a strictly increasing function. We will also see that the utilities that are of interest to advantage players are concave down. Given a wager with payoffs of $\langle X_1, X_2, \ldots\rangle$ per unit bet and corresponding probabilities $\langle p_1, p_2, \ldots\rangle$ we can compute its expected utility $\mathrm{E}U = \sum p_k U(B + X_k b)$. Here B is our wealth prior to the wager, and b is the bet size. The certainty equivalent is the quantity CE that satisfies

$$U(B + \mathrm{CE}) = \mathrm{E}U = \sum p_k U(B + X_k b). \tag{42}$$

That is, if we had a “wager” that paid us the amount CE with probability 100%, then it would produce the same expected utility as EU. Equation (42) does define a function; since U is strictly increasing, it has an inverse U^{-1} and we could explicitly express CE as

$$\mathrm{CE} = U^{-1}(\mathrm{E}U) - B = U^{-1}\left(\sum p_k U(B + X_k b)\right) - B. \tag{43}$$

However, I find it more convenient to work from (42). We will treat CE as a function CE(b) of the bet size b. As such, it has a second-order Taylor expansion of the form

$$\mathrm{CE}(b) = \mathrm{CE}(0) + \mathrm{CE}'(0)b + \frac{1}{2}\,\mathrm{CE}''(0)b^2 + O(b^3). \tag{44}$$

CE(0) is clearly 0, and we can find the derivatives CE' and CE'' from (42), using implicit differentiation. Taking the derivative of both sides gives us

$$U'(B + \mathrm{CE})\,\mathrm{CE}' = \sum p_k U'(B + X_k b) X_k, \tag{45}$$

from which we obtain

$$\mathrm{CE}'(b) = \sum p_k U'(B + X_k b) X_k / U'(B + \mathrm{CE}(b)). \tag{46}$$

This gives us that $\mathrm{CE}'(0) = \sum p_k U'(B) X_k / U'(B) = \mu$. Taking the second derivative with the quotient rule gives us

$$\begin{aligned}\mathrm{CE}''(b) = \Big\{ \sum p_k U''(B + X_k b) X_k^2 U'(B + \mathrm{CE}(b)) \\ - \sum p_k U'(B + X_k b) X_k U''(B + \mathrm{CE}(b))\, \mathrm{CE}'(b) \Big\} \\ \Big/ [U'(B + \mathrm{CE}(b))]^2.\end{aligned} \tag{47}$$

Fortunately, this simplifies when we substitute in $b = 0$. Remembering that $\mathrm{CE}(0) = 0$, we have

$$\begin{aligned}\mathrm{CE}''(0) = \Big\{ \sum p_k U''(B) X_k^2 U'(B) - \sum p_k U'(B) X_k U''(B)\, \mathrm{CE}'(0) \Big\} \\ \Big/ [U'(B)]^2.\end{aligned} \tag{48}$$

Now $\sum p_k X_k^2$ is the second moment M_2 of our wager and $\sum p_k X_k$ is the mean μ, and we know that $\mathrm{CE}'(0) = \mu$. Substituting, we have

$$\mathrm{CE}''(0) = \frac{U''(B)}{U'(B)} (M_2 - \mu^2) = \frac{U''(B)}{U'(B)} \sigma^2. \tag{49}$$

For the utilities that interest us, $U''(B) < 0$ and $U'(B) > 0$, so the coefficient of σ^2 is negative. For some of the interesting utilities described below, the magnitude of this coefficient is inversely proportional to B. This motivates the definition

$$k = -U'(B)/[BU''(B)], \tag{50}$$

which happens to be the reciprocal of the Arrow–Pratt measure of relative risk aversion, so that our coefficient of σ^2 is $-1/(kB)$. Eq. (44) now becomes

$$\mathrm{CE} = \mu b - \frac{\sigma^2}{2kB}\, b^2 + O(b^3). \tag{51}$$

The first term μb is the expected value of the wager, and $\sigma^2 b^2$ is the variance. With slightly different notation, we obtain eq. (1).

I conclude this section with the promised discussion of some specific utility functions. The Kelly betting system is associated with logarithmic utility $U(B) = \ln B$; Kelly players seek to maximize the expected logarithm of the bankroll $\mathrm{E}[\ln B]$. For this utility, $k = -U'/[BU''] = -(B^{-1})/[B(-B^{-2})] = 1$, as discussed in Section 1.

If we wished to determine the value of b that maximizes our CE, we could approximate this by finding where our quadratic approximation reaches a maximum. This occurs at the vertex:

$$b_{\text{opt}} \approx \frac{\mu}{\sigma^2}\, kB, \tag{52}$$

which is the classical approximation for optimal betting.

Another interesting utility function is $U(B) = -B^{-1}$. For this utility, $k = -(B^{-2})/(-2B^{-3}B) = 1/2$. Players using this utility would thus bet half of what a Kelly player would in any given situation; this is called half-Kelly betting. Here 1/2 is sometimes called the "Kelly fraction." We can obtain utilities for any other Kelly fraction $k \in (0, 1)$, by selecting $U(W) = -W^p$, where $p = (k-1)/k$.

At the beginning of this section I stated that the utilities of interest to advantage players are concave down. For such utilities, our k will be positive, and the coefficient of σ^2 in (51) will be negative. Then even negative-expectation wagers would have positive CE. That is, such utilities would call for making negative-expectation wagers, which is not characteristic of advantage gamblers.

Appendix 3. Graphs of partial insurance

The graphs illustrate the interval of partial insurance. The first example is a mediocre hand (Hard 18) with big bet ($f = 5\%$).

The length of the interval is proportional to the bet size f. The μ-length is f. The d-length is $f/3$. The Hi-Lo length is approximately $f/2.3\%$ Hi-Lo points.

The conventional index ($f = 0$) occurs at $d = 1/3$. It is usually within the interval. For weak hands, it is to the left of the interval. For strong hands, it is to the right of the interval. Exceptions: For BJ, the conventional index is the right endpoint. For 20, the conventional index is outside the interval.

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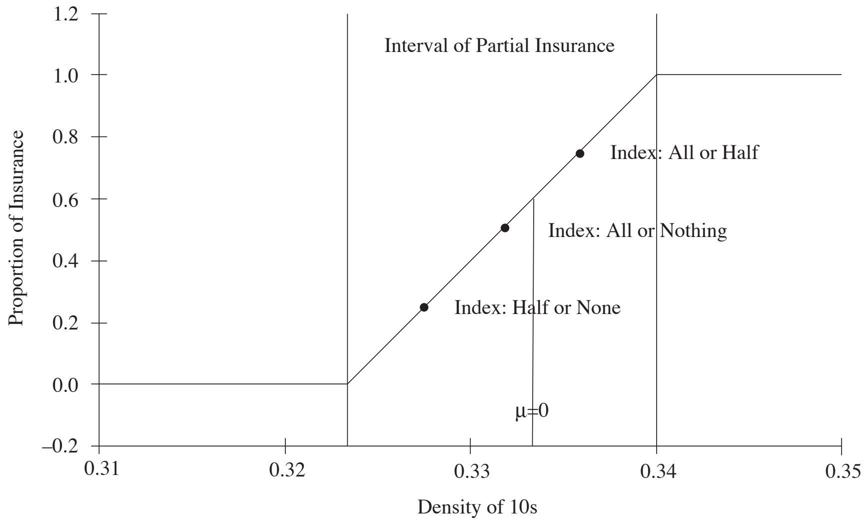

Fig. 1. Graph for player 18.

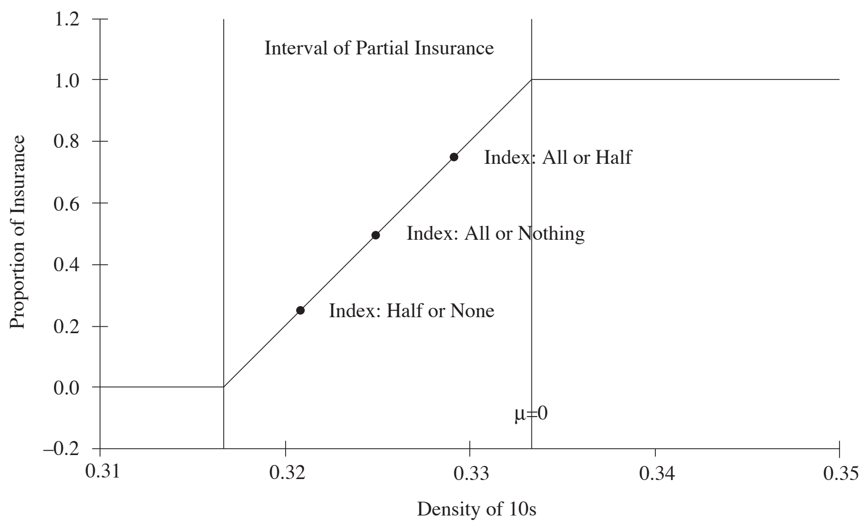

Fig. 2. Graph for player 21.

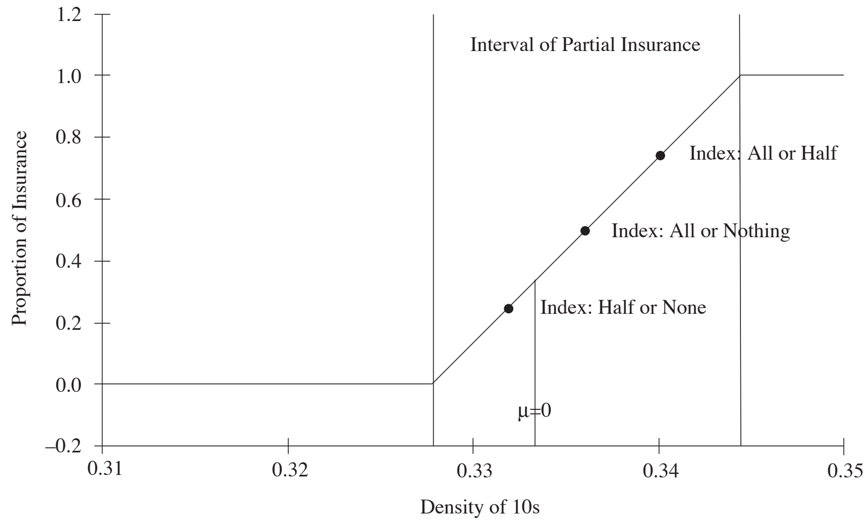

Fig. 3. Graph for player Surrender.

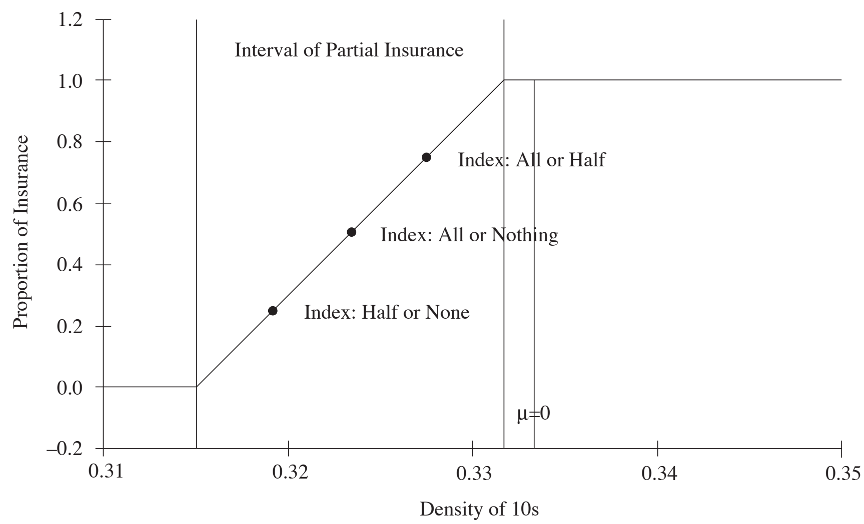

Fig. 4. Graph for player 20.

Table A1. Minimizing variance with p_v. Optimal partial insurance at the break-even point ($\mu = 0$). Infinite deck; dealer stands on soft 17. Note: Conditional expectations were computed using an infinite deck, with additional 10s added to make 10s-density equal 1/3.

hand	action	CondEV	part. ins. p_v	ins. bet
Any 20	Stand	66%	111%	0.55
BJ	Stand	150%	100%	0.50
Any 19	Stand	29%	86%	0.43
A, A	Split	20%	80%	0.40
Hard 11	Double	20%	80%	0.40
Hard 10	Hit	10%	73%	0.37
Hard 9	Hit	−5%	63%	0.32
Any 18	Stand	−8%	61%	0.31
Soft 13	Hit	−8%	61%	0.31
Soft 14	Hit	−12%	59%	0.29
Soft 15	Hit	−15%	56%	0.28
Soft 17	Hit	−19%	54%	0.27
Soft 16	Hit	−19%	54%	0.27
Hard 8	Hit	−19%	54%	0.27
Hard 4	Hit	−27%	49%	0.24
Hard 5	Hit	−29%	47%	0.24
Hard 7	Hit	−31%	46%	0.23
Hard 6	Hit	−32%	45%	0.23
8, 8	Split	−35%	43%	0.22
Hard 12	Hit	−38%	42%	0.21
Hard 13	Hit	−42%	39%	0.19
Hard 17	Stand	−46%	36%	0.18
Hard 14	Hit	−46%	36%	0.18
Hard 15	Hit	−50%	34%	0.17
Hard 16	Surrender	−50%	33%	0.17
Hard 16	Hit	−53%	31%	0.16

Table A2. Risk-averse insurance interval and indices. Infinite deck; unbalanced 10 count. Conventional index = 0. Bet size = 2%.

hand	action	hand strength		interval		indices		
		CondEV	p_v	left	right	all/none	half/none	half/all
Any 20	Stand	66.30%	110.8%	−1.15	−0.11	−0.63	−0.89	−0.37
BJ	Stand	150.00%	100.0%	−1.04	0.00	−0.52	−0.78	−0.26
Any 19	Stand	29.00%	86.0%	−0.89	0.15	−0.37	−0.63	−0.11
A, A	Split	20.20%	80.2%	−0.83	0.21	−0.31	−0.57	−0.05
Hard 11	Dble	20.20%	80.2%	−0.83	0.21	−0.31	−0.57	−0.05
Hard 10	Hit	10.20%	73.4%	−0.76	0.28	−0.24	−0.50	0.02
Hard 9	Hit	−5.00%	63.4%	−0.66	0.38	−0.14	−0.40	0.12
Any 18	Stand	−8.30%	61.2%	−0.64	0.40	−0.12	−0.38	0.14
Soft 13	Hit	−8.20%	61.2%	−0.64	0.40	−0.12	−0.38	0.14
Soft 14	Hit	−11.80%	58.8%	−0.61	0.43	−0.09	−0.35	0.17
Soft 15	Hit	−15.30%	56.4%	−0.59	0.45	−0.07	−0.33	0.19
Soft 17	Hit	−18.80%	54.2%	−0.56	0.48	−0.04	−0.30	0.22
Soft 16	Hit	−18.70%	54.2%	−0.56	0.48	−0.04	−0.30	0.22
Hard 8	Hit	−18.90%	54.0%	−0.56	0.48	−0.04	−0.30	0.22
Hard 4	Hit	−26.70%	48.8%	−0.51	0.53	0.01	−0.25	0.27
Hard 5	Hit	−29.20%	47.2%	−0.49	0.55	0.03	−0.23	0.29
Hard 7	Hit	−31.00%	46.0%	−0.48	0.56	0.04	−0.22	0.30
Hard 6	Hit	−31.80%	45.4%	−0.47	0.57	0.05	−0.21	0.31
8, 8	Split	−34.90%	43.4%	−0.45	0.59	0.07	−0.19	0.33
Hard 12	Hit	−37.50%	41.6%	−0.43	0.61	0.09	−0.17	0.35
Hard 13	Hit	−41.80%	38.8%	−0.40	0.64	0.12	−0.14	0.38
Hard 17	Stand	−45.60%	36.2%	−0.38	0.66	0.14	−0.12	0.40
Hard 14	Hit	−45.80%	36.2%	−0.38	0.66	0.14	−0.12	0.40
Hard 15	Hit	−49.60%	33.6%	−0.35	0.69	0.17	−0.09	0.43
Hard 16	Surrender	−50.00%	33.3%	−0.35	0.69	0.17	−0.09	0.43
Hard 16	Hit	−53.00%	31.4%	−0.33	0.71	0.19	−0.07	0.45

Table A3. Risk-averse insurance interval and indices. Infinite deck; unbalanced 10 count. Conventional index = 0. Bet size = 5%.

hand	action	hand strength		interval		indices		
		CondEV	p_v	left	right	all/none	half/none	half/all
Any 20	Stand	66.30%	110.8%	−2.88	−0.28	−1.58	−2.23	−0.93
BJ	Stand	150.00%	100.0%	−2.60	0.00	−1.30	−1.95	−0.65
Any 19	Stand	29.00%	86.0%	−2.24	0.36	−0.94	−1.59	−0.29
A, A	Split	20.20%	80.2%	−2.09	0.51	−0.79	−1.44	−0.14
Hard 11	Dble	20.20%	80.2%	−2.09	0.51	−0.79	−1.44	−0.14
Hard 10	Hit	10.20%	73.4%	−1.91	0.69	−0.61	−1.26	0.04
Hard 9	Hit	−5.00%	63.4%	−1.65	0.95	−0.35	−1.00	0.30
Any 18	Stand	−8.30%	61.2%	−1.59	1.01	−0.29	−0.94	0.36
Soft 13	Hit	−8.20%	61.2%	−1.59	1.01	−0.29	−0.94	0.36
Soft 14	Hit	−11.80%	58.8%	−1.53	1.07	−0.23	−0.88	0.42
Soft 15	Hit	−15.30%	56.4%	−1.47	1.13	−0.17	−0.82	0.48
Soft 17	Hit	−18.80%	54.2%	−1.41	1.19	−0.11	−0.76	0.54
Soft 16	Hit	−18.70%	54.2%	−1.41	1.19	−0.11	−0.76	0.54
Hard 8	Hit	−18.90%	54.0%	−1.40	1.20	−0.10	−0.75	0.55
Hard 4	Hit	−26.70%	48.8%	−1.27	1.33	0.03	−0.62	0.68
Hard 5	Hit	−29.20%	47.2%	−1.23	1.37	0.07	−0.58	0.72
Hard 7	Hit	−31.00%	46.0%	−1.20	1.40	0.10	−0.55	0.75
Hard 6	Hit	−31.80%	45.4%	−1.18	1.42	0.12	−0.53	0.77
8, 8	Split	−34.90%	43.4%	−1.13	1.47	0.17	−0.48	0.82
Hard 12	Hit	−37.50%	41.6%	−1.08	1.52	0.22	−0.43	0.87
Hard 13	Hit	−41.80%	38.8%	−1.01	1.59	0.29	−0.36	0.94
Hard 17	Stand	−45.60%	36.2%	−0.94	1.66	0.36	−0.29	1.01
Hard 14	Hit	−45.80%	36.2%	−0.94	1.66	0.36	−0.29	1.01
Hard 15	Hit	−49.60%	33.6%	−0.87	1.73	0.43	−0.22	1.08
Hard 16	Surrender	−50.00%	33.3%	−0.87	1.73	0.43	−0.22	1.08
Hard 16	Hit	−53.00%	31.4%	−0.82	1.78	0.48	−0.17	1.13

Table A4. Hi-Lo all/nothing insurance: Index shifts. 1 deck (H17). Difference between generic index (1.4) and RA/CD-Index.

hand		composition adjustment	hand strength CondEV	p_v	bet size 1%	2%	5%
AA	Soft 12	−3.84	0.236	0.824	−4.0	−4.1	−4.5
AX	BJ	−1.15	1.500	1.000	−1.4	−1.6	−2.2
A9	Soft 20	−2.71	0.636	1.091	−3.0	−3.2	−4.0
A8	Soft 19	−2.71	0.205	0.803	−2.8	−3.0	−3.4
A7	Soft 18	−2.71	−0.173	0.552	−2.7	−2.8	−2.8
A6	Soft 17	−1.71	−0.238	0.508	−1.7	−1.7	−1.7
A5	Soft 16	−1.71	−0.268	0.488	−1.7	−1.7	−1.7
A4	Soft 15	−1.71	−0.185	0.543	−1.7	−1.7	−1.8
A3	Soft 14	−1.71	−0.147	0.569	−1.7	−1.8	−1.9
A2	Soft 13	−1.71	−0.091	0.606	−1.8	−1.8	−1.9
XX	Hard 20	1.54	0.602	1.068	1.3	1.0	0.3
X9	Hard 19	−0.02	0.232	0.821	−0.2	−0.3	−0.7
X8	Hard 18	−0.02	−0.184	0.544	0.0	−0.1	−0.1
X7	Hard 17	−0.02	−0.476	0.349	0.0	0.1	0.3
X6	Hard 16	0.98	−0.542	0.305	1.1	1.1	1.4
X5	Hard 15	0.98	−0.537	0.308	1.1	1.1	1.4
X4	Hard 14	0.98	−0.486	0.343	1.0	1.1	1.3
X3	Hard 13	0.98	−0.440	0.373	1.0	1.1	1.3
X2	Hard 12	0.98	−0.402	0.398	1.0	1.1	1.2
99	Hard 18	−1.58	−0.185	0.544	−1.6	−1.6	−1.7
98	Hard 17	−1.58	−0.483	0.345	−1.5	−1.4	−1.2
97	Hard 16	−1.58	−0.548	0.301	−1.5	−1.4	−1.1
96	Hard 15	−0.58	−0.518	0.321	−0.5	−0.4	−0.2
95	Hard 14	−0.58	−0.480	0.347	−0.5	−0.4	−0.2
94	Hard 13	−0.58	−0.399	0.400	−0.5	−0.5	−0.4
93	Hard 12	−0.58	−0.405	0.396	−0.5	−0.5	−0.4
92	Hard 11	−0.58	0.207	0.805	−0.7	−0.8	−1.2
88	Hard 16	−1.58	−0.473	0.352	−1.5	−1.5	−1.3
87	Hard 15	−1.58	−0.485	0.343	−1.5	−1.4	−1.2
86	Hard 14	−0.58	−0.487	0.342	−0.5	−0.4	−0.2
85	Hard 13	−0.58	−0.445	0.370	−0.5	−0.5	−0.3
84	Hard 12	−0.58	−0.403	0.398	−0.5	−0.5	−0.4
83	Hard 11	−0.58	0.226	0.817	−0.7	−0.9	−1.3
82	Hard 10	−0.58	0.041	0.694	−0.7	−0.7	−1.0
77	Hard 14	−1.58	−0.530	0.314	−1.5	−1.4	−1.2
76	Hard 13	−0.58	−0.451	0.366	−0.5	−0.5	−0.3
75	Hard 12	−0.58	−0.408	0.395	−0.5	−0.5	−0.4
74	Hard 11	−0.58	0.245	0.830	−0.7	−0.9	−1.3
73	Hard 10	−0.58	0.040	0.694	−0.7	−0.7	−1.0
72	Hard 9	−0.58	−0.133	0.578	−0.6	−0.6	−0.7
66	Hard 12	0.42	−0.405	0.396	0.5	0.5	0.6
65	Hard 11	0.42	0.252	0.834	0.3	0.1	−0.3
64	Hard 10	0.42	0.046	0.698	0.3	0.2	0.0
63	Hard 9	0.42	−0.119	0.587	0.4	0.3	0.2
62	Hard 8	0.42	−0.278	0.482	0.4	0.4	0.5
55	Hard 10	0.42	0.042	0.695	0.3	0.3	0.0
54	Hard 9	0.42	−0.120	0.587	0.4	0.3	0.2
53	Hard 8	0.42	−0.281	0.479	0.4	0.4	0.5
52	Hard 7	0.42	−0.368	0.422	0.5	0.5	0.6
44	Hard 8	0.42	−0.269	0.487	0.4	0.4	0.4
43	Hard 7	0.42	−0.377	0.415	0.5	0.5	0.6
42	Hard 6	0.42	−0.378	0.414	0.5	0.5	0.6
33	Hard 6	0.42	−0.379	0.414	0.5	0.5	0.6
32	Hard 5	0.42	−0.323	0.451	0.4	0.5	0.5
22	Hard 4	0.42	−0.298	0.468	0.4	0.4	0.5

Table A5. Hi-Lo all/nothing insurance: Index shifts. 6 decks (S17-DAS). Difference between generic index (3.0) and RA/CD-Index.

hand		composition adjustment	hand strength CondEV	p_v	bet size 1%	2%	5%
AA	Soft 12	−0.64	0.21	0.80	−0.8	−0.9	−1.3
AX	BJ	−0.19	1.50	1.00	−0.4	−0.6	−1.3
A9	Soft 20	−0.45	0.67	1.11	−0.7	−1.0	−1.8
A8	Soft 19	−0.45	0.29	0.86	−0.6	−0.8	−1.2
A7	Soft 18	−0.45	−0.09	0.61	−0.5	−0.5	−0.7
A6	Soft 17	−0.28	−0.19	0.54	−0.3	−0.3	−0.4
A5	Soft 16	−0.28	−0.19	0.54	−0.3	−0.3	−0.4
A4	Soft 15	−0.28	−0.15	0.57	−0.3	−0.3	−0.4
A3	Soft 14	−0.28	−0.12	0.59	−0.3	−0.4	−0.5
A2	Soft 13	−0.28	−0.08	0.62	−0.3	−0.4	−0.5
XX	Hard 20	0.26	0.66	1.11	0.0	−0.3	−1.1
X9	Hard 19	0.00	0.30	0.86	−0.2	−0.3	−0.8
X8	Hard 18	0.00	−0.08	0.61	0.0	−0.1	−0.2
X7	Hard 17	0.00	−0.46	0.36	0.1	0.1	0.3
X6	Hard 16	0.16	−0.53	0.31	0.2	0.3	0.6
X5	Hard 15	0.16	−0.50	0.34	0.2	0.3	0.5
X4	Hard 14	0.16	−0.46	0.36	0.2	0.3	0.5
X3	Hard 13	0.16	−0.42	0.39	0.2	0.3	0.4
X2	Hard 12	0.16	−0.37	0.42	0.2	0.2	0.3
99	Hard 18	−0.26	−0.08	0.62	−0.3	−0.4	−0.5
98	Hard 17	−0.26	−0.46	0.36	−0.2	−0.1	0.0
97	Hard 16	−0.26	−0.53	0.31	−0.2	−0.1	0.1
96	Hard 15	−0.10	−0.50	0.34	0.0	0.0	0.3
95	Hard 14	−0.10	−0.46	0.36	0.0	0.0	0.2
94	Hard 13	−0.10	−0.41	0.39	−0.1	0.0	0.1
93	Hard 12	−0.10	−0.37	0.42	−0.1	0.0	0.1
92	Hard 11	−0.10	0.20	0.80	−0.2	−0.4	−0.8
88	Hard 16	−0.26	−0.34	0.44	−0.2	−0.2	−0.1
87	Hard 15	−0.26	−0.49	0.34	−0.2	−0.1	0.1
86	Hard 14	−0.10	−0.46	0.36	0.0	0.0	0.2
85	Hard 13	−0.10	−0.42	0.39	−0.1	0.0	0.1
84	Hard 12	−0.10	−0.37	0.42	−0.1	0.0	0.1
83	Hard 11	−0.10	0.21	0.80	−0.2	−0.4	−0.8
82	Hard 10	−0.10	0.10	0.73	−0.2	−0.3	−0.6
77	Hard 14	−0.26	−0.47	0.36	−0.2	−0.1	0.0
76	Hard 13	−0.10	−0.42	0.39	−0.1	0.0	0.1
75	Hard 12	−0.10	−0.37	0.42	−0.1	0.0	0.1
74	Hard 11	−0.10	0.21	0.81	−0.2	−0.4	−0.8
73	Hard 10	−0.10	0.10	0.73	−0.2	−0.3	−0.6
72	Hard 9	−0.10	−0.05	0.63	−0.2	−0.2	−0.4
66	Hard 12	0.07	−0.38	0.42	0.1	0.1	0.2
65	Hard 11	0.07	0.21	0.81	−0.1	−0.2	−0.6
64	Hard 10	0.07	0.10	0.73	0.0	−0.1	−0.4
63	Hard 9	0.07	−0.05	0.63	0.0	0.0	−0.2
62	Hard 8	0.07	−0.19	0.54	0.1	0.0	0.0
55	Hard 10	0.07	0.10	0.73	0.0	−0.1	−0.4
54	Hard 9	0.07	−0.05	0.63	0.0	0.0	−0.2
53	Hard 8	0.07	−0.19	0.54	0.1	0.0	0.0
52	Hard 7	0.07	−0.31	0.46	0.1	0.1	0.2
44	Hard 8	0.07	−0.19	0.54	0.1	0.0	0.0
43	Hard 7	0.07	−0.32	0.46	0.1	0.1	0.2
42	Hard 6	0.07	−0.32	0.45	0.1	0.1	0.2
33	Hard 6	0.07	−0.32	0.45	0.1	0.1	0.2
32	Hard 5	0.07	−0.29	0.47	0.1	0.1	0.1
22	Hard 4	0.07	−0.27	0.49	0.1	0.1	0.1

Table A6. Complete Hi-Lo single-deck insurance indices. Bet size: 1%. Rules: H17.

hand		composition adjustment	hand strength CondEV	p_v	interval left	right	indices all/none	half/none	half/all
AA	Pair Aces	−3.8	0.24	0.8	−2.8	−2.3	−2.5	−2.7	−2.4
AX	BJ	−1.2	1.50	1.0	−0.2	0.3	0.1	0.0	0.2
A9	Soft 20	−2.7	0.64	1.1	−1.8	−1.3	−1.5	−1.6	−1.4
A8	Soft 19	−2.7	0.21	0.8	−1.6	−1.2	−1.4	−1.5	−1.3
A7	Soft 18	−2.7	−0.17	0.6	−1.5	−1.1	−1.3	−1.4	−1.2
A6	Soft 17	−1.7	−0.24	0.5	−0.5	−0.1	−0.3	−0.4	−0.2
A5	Soft 16	−1.7	−0.27	0.5	−0.5	−0.1	−0.3	−0.4	−0.2
A4	Soft 15	−1.7	−0.19	0.5	−0.5	−0.1	−0.3	−0.4	−0.2
A3	Soft 14	−1.7	−0.15	0.6	−0.5	−0.1	−0.3	−0.4	−0.2
A2	Soft 13	−1.7	−0.09	0.6	−0.5	−0.1	−0.3	−0.4	−0.2
XX	Hard 20	1.5	0.60	1.1	2.5	2.9	2.7	2.6	2.8
X9	Hard 19	0.0	0.23	0.8	1.1	1.5	1.3	1.2	1.4
X8	Hard 18	0.0	−0.18	0.5	1.2	1.6	1.4	1.3	1.5
X7	Hard 17	0.0	−0.48	0.3	1.3	1.7	1.5	1.4	1.6
X6	Hard 16	1.0	−0.54	0.3	2.3	2.7	2.5	2.4	2.6
X5	Hard 15	1.0	−0.54	0.3	2.3	2.7	2.5	2.4	2.6
X4	Hard 14	1.0	−0.49	0.3	2.3	2.7	2.5	2.4	2.6
X3	Hard 13	1.0	−0.44	0.4	2.3	2.7	2.5	2.4	2.6
X2	Hard 12	1.0	−0.40	0.4	2.2	2.7	2.5	2.3	2.6
99	Hard 18	−1.6	−0.19	0.5	−0.4	0.1	−0.2	−0.3	−0.1
98	Hard 17	−1.6	−0.48	0.3	−0.3	0.1	−0.1	−0.2	0.0
97	Hard 16	−1.6	−0.55	0.3	−0.3	0.2	−0.1	−0.2	0.0
96	Hard 15	−0.6	−0.52	0.3	0.7	1.1	0.9	0.8	1.0
95	Hard 14	−0.6	−0.48	0.3	0.7	1.1	0.9	0.8	1.0
94	Hard 13	−0.6	−0.40	0.4	0.7	1.1	0.9	0.8	1.0
93	Hard 12	−0.6	−0.41	0.4	0.7	1.1	0.9	0.8	1.0
92	Hard 11	−0.6	0.21	0.8	0.5	0.9	0.7	0.6	0.8
88	Pair 8s	−1.6	−0.47	0.4	−0.3	0.1	−0.1	−0.2	0.0
87	Hard 15	−1.6	−0.49	0.3	−0.3	0.1	−0.1	−0.2	0.0
86	Hard 14	−0.6	−0.49	0.3	0.7	1.1	0.9	0.8	1.0
85	Hard 13	−0.6	−0.45	0.4	0.7	1.1	0.9	0.8	1.0
84	Hard 12	−0.6	−0.40	0.4	0.7	1.1	0.9	0.8	1.0
83	Hard 11	−0.6	0.23	0.8	0.5	0.9	0.7	0.6	0.8
82	Hard 10	−0.6	0.04	0.7	0.6	1.0	0.8	0.7	0.9
77	Hard 14	−1.6	−0.53	0.3	−0.3	0.2	−0.1	−0.2	0.0
76	Hard 13	−0.6	−0.45	0.4	0.7	1.1	0.9	0.8	1.0
75	Hard 12	−0.6	−0.41	0.4	0.7	1.1	0.9	0.8	1.0
74	Hard 11	−0.6	0.25	0.8	0.5	0.9	0.7	0.6	0.8
73	Hard 10	−0.6	0.04	0.7	0.6	1.0	0.8	0.7	0.9
72	Hard 9	−0.6	−0.13	0.6	0.6	1.0	0.8	0.7	0.9
66	Hard 12	0.4	−0.41	0.4	1.7	2.1	1.9	1.8	2.0
65	Hard 11	0.4	0.25	0.8	1.5	1.9	1.7	1.6	1.8
64	Hard 10	0.4	0.05	0.7	1.5	2.0	1.8	1.7	1.9
63	Hard 9	0.4	−0.12	0.6	1.6	2.0	1.8	1.7	1.9
62	Hard 8	0.4	−0.28	0.5	1.6	2.1	1.9	1.8	2.0
55	Hard 10	0.4	0.04	0.7	1.6	2.0	1.8	1.7	1.9
54	Hard 9	0.4	−0.12	0.6	1.6	2.0	1.8	1.7	1.9
53	Hard 8	0.4	−0.28	0.5	1.6	2.1	1.9	1.8	2.0
52	Hard 7	0.4	−0.37	0.4	1.7	2.1	1.9	1.8	2.0
44	Hard 8	0.4	−0.27	0.5	1.6	2.1	1.9	1.8	2.0
43	Hard 7	0.4	−0.38	0.4	1.7	2.1	1.9	1.8	2.0
42	Hard 6	0.4	−0.38	0.4	1.7	2.1	1.9	1.8	2.0
33	Hard 6	0.4	−0.38	0.4	1.7	2.1	1.9	1.8	2.0
32	Hard 5	0.4	−0.32	0.5	1.7	2.1	1.9	1.8	2.0
22	Hard 4	0.4	−0.30	0.5	1.6	2.1	1.9	1.8	2.0

Table A7. Complete Hi-Lo six-deck insurance indices. Bet size: 4%. Rules: S17 DAS.

hand		composition adjustment	hand strength CondEV	p_v	interval left	right	indices all/none	half/none	half/all
AA	Pair Aces	−0.64	0.21	0.80	1.0	2.7	1.8	1.4	2.3
AX	BJ	−0.19	1.50	1.00	1.1	2.8	2.0	1.5	2.4
A9	Soft 20	−0.45	0.67	1.11	0.6	2.4	1.5	1.1	1.9
A8	Soft 19	−0.45	0.29	0.86	1.1	2.8	1.9	1.5	2.4
A7	Soft 18	−0.45	−0.09	0.61	1.5	3.2	2.4	1.9	2.8
A6	Soft 17	−0.28	−0.19	0.54	1.8	3.5	2.7	2.2	3.1
A5	Soft 16	−0.28	−0.19	0.54	1.8	3.5	2.7	2.2	3.1
A4	Soft 15	−0.28	−0.15	0.57	1.8	3.5	2.6	2.2	3.1
A3	Soft 14	−0.28	−0.12	0.59	1.7	3.5	2.6	2.1	3.0
A2	Soft 13	−0.28	−0.08	0.62	1.7	3.4	2.5	2.1	3.0
XX	Hard 20	0.26	0.66	1.11	1.4	3.1	2.2	1.8	2.7
X9	Hard 19	0.00	0.30	0.86	1.5	3.3	2.4	2.0	2.8
X8	Hard 18	0.00	−0.08	0.61	2.0	3.7	2.8	2.4	3.3
X7	Hard 17	0.00	−0.46	0.36	2.4	4.1	3.3	2.8	3.7
X6	Hard 16	0.16	−0.53	0.31	2.6	4.4	3.5	3.1	3.9
X5	Hard 15	0.16	−0.50	0.34	2.6	4.3	3.5	3.0	3.9
X4	Hard 14	0.16	−0.46	0.36	2.5	4.3	3.4	3.0	3.9
X3	Hard 13	0.16	−0.42	0.39	2.5	4.2	3.4	2.9	3.8
X2	Hard 12	0.16	−0.37	0.42	2.4	4.2	3.3	2.9	3.8
99	Hard 18	−0.26	−0.08	0.62	1.7	3.4	2.6	2.1	3.0
98	Hard 17	−0.26	−0.46	0.36	2.1	3.9	3.0	2.6	3.4
97	Hard 16	−0.26	−0.53	0.31	2.2	3.9	3.1	2.6	3.5
96	Hard 15	−0.10	−0.50	0.34	2.3	4.1	3.2	2.8	3.6
95	Hard 14	−0.10	−0.46	0.36	2.3	4.0	3.2	2.7	3.6
94	Hard 13	−0.10	−0.41	0.39	2.2	4.0	3.1	2.7	3.5
93	Hard 12	−0.10	−0.37	0.42	2.2	3.9	3.1	2.6	3.5
92	Hard 11	−0.10	0.20	0.80	1.5	3.3	2.4	2.0	2.8
88	Pair 8s	−0.26	−0.34	0.44	2.0	3.7	2.9	2.4	3.3
87	Hard 15	−0.26	−0.49	0.34	2.2	3.9	3.0	2.6	3.5
86	Hard 14	−0.10	−0.46	0.36	2.3	4.0	3.2	2.7	3.6
85	Hard 13	−0.10	−0.42	0.39	2.2	4.0	3.1	2.7	3.5
84	Hard 12	−0.10	−0.37	0.42	2.2	3.9	3.1	2.6	3.5
83	Hard 11	−0.10	0.21	0.80	1.5	3.3	2.4	2.0	2.8
82	Hard 10	−0.10	0.10	0.73	1.6	3.4	2.5	2.1	2.9
77	Hard 14	−0.26	−0.47	0.36	2.1	3.9	3.0	2.6	3.4
76	Hard 13	−0.10	−0.42	0.39	2.2	4.0	3.1	2.7	3.5
75	Hard 12	−0.10	−0.37	0.42	2.2	3.9	3.1	2.6	3.5
74	Hard 11	−0.10	0.21	0.81	1.5	3.3	2.4	2.0	2.8
73	Hard 10	−0.10	0.10	0.73	1.6	3.4	2.5	2.1	2.9
72	Hard 9	−0.10	−0.05	0.63	1.8	3.6	2.7	2.3	3.1
66	Hard 12	0.07	−0.38	0.42	2.4	4.1	3.2	2.8	3.7
65	Hard 11	0.07	0.21	0.81	1.7	3.4	2.6	2.1	3.0
64	Hard 10	0.07	0.10	0.73	1.8	3.6	2.7	2.2	3.1
63	Hard 9	0.07	−0.05	0.63	2.0	3.7	2.9	2.4	3.3
62	Hard 8	0.07	−0.19	0.54	2.2	3.9	3.0	2.6	3.5
55	Hard 10	0.07	0.10	0.73	1.8	3.6	2.7	2.2	3.1
54	Hard 9	0.07	−0.05	0.63	2.0	3.7	2.9	2.4	3.3
53	Hard 8	0.07	−0.19	0.54	2.2	3.9	3.0	2.6	3.5
52	Hard 7	0.07	−0.31	0.46	2.3	4.0	3.2	2.7	3.6
44	Hard 8	0.07	−0.19	0.54	2.1	3.9	3.0	2.6	3.5
43	Hard 7	0.07	−0.32	0.46	2.3	4.0	3.2	2.7	3.6
42	Hard 6	0.07	−0.32	0.45	2.3	4.0	3.2	2.7	3.6
33	Hard 6	0.07	−0.32	0.45	2.3	4.0	3.2	2.7	3.6
32	Hard 5	0.07	−0.29	0.47	2.3	4.0	3.1	2.7	3.6
22	Hard 4	0.07	−0.27	0.49	2.2	4.0	3.1	2.7	3.5